An adaptive and conservative low-rank IMEX solver for the hybrid ion Vlasov-Fokker-Planck (VFP) and fluid electron system

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1 Motivation

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Kinetic models and high-dimensional PDEs

Kinetic models appear across a wide array of applications, from high-altitude flight aerodynamic simulations to population dynamics to ionized plasma modeling for ionic propulsion and nuclear fusion.

$$f_t + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathbf{a} \cdot \nabla_{\mathbf{v}} f = C(f) \tag{1}$$

where $C(f)$ is a collision operator (e.g. BGK, Boltzmann, Fokker-Planck).

- Solve for distribution function $f(\mathbf{x}, \mathbf{v}, t) : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$.

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- High-dimensional PDEs suffer from the **curse of dimensionality**
  - Naively storing $\mathcal{U}(t) \in \mathbb{R}^{N \times \ldots \times N}$ requires $\mathcal{O}(N^d)$ degrees of freedom.

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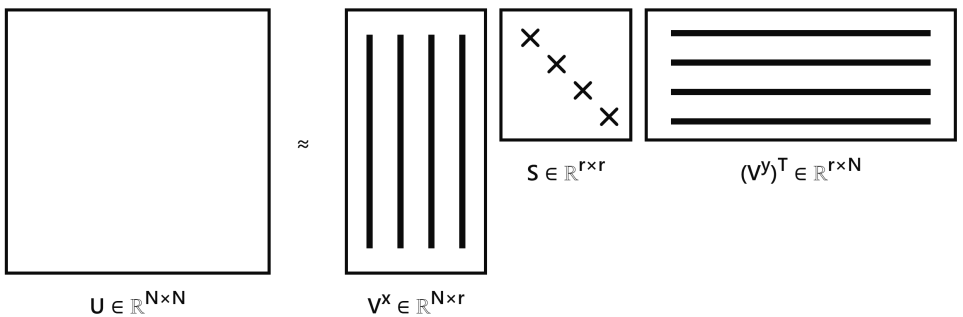
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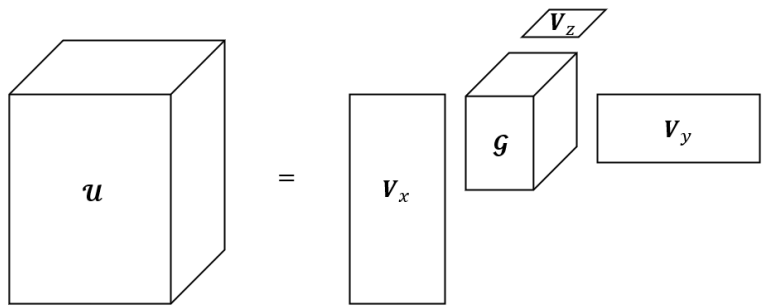
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- Multi-scale
- Conservation of macroscopic quantities (e.g. mass, momentum, energy)
- Preservation of physically correct equilibrium solutions

Low-rank tensor decompositions

Many PDE solutions exhibit low-rank structure ($r \ll N$) that we can leverage with tensor decompositions.



(a) SVD: $\mathcal{O}(N^2) \longrightarrow \mathcal{O}(2Nr + r^2)$



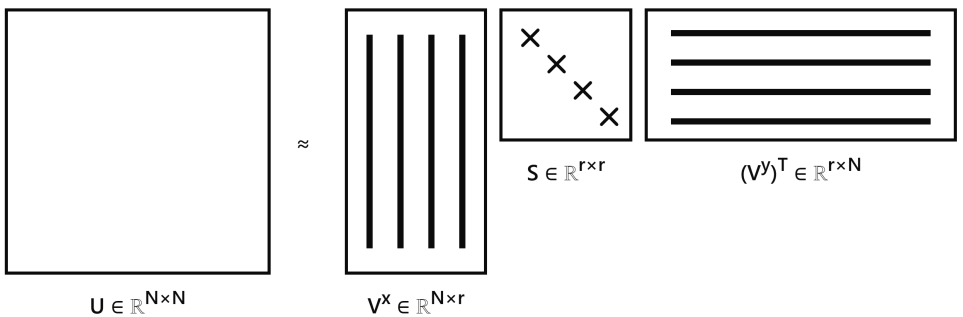
(b) Tucker: $\mathcal{O}(N^d) \longrightarrow \mathcal{O}(dNr + r^d)$

Tucker, hierarchical Tucker, tensor train, tree tensor networks, etc...

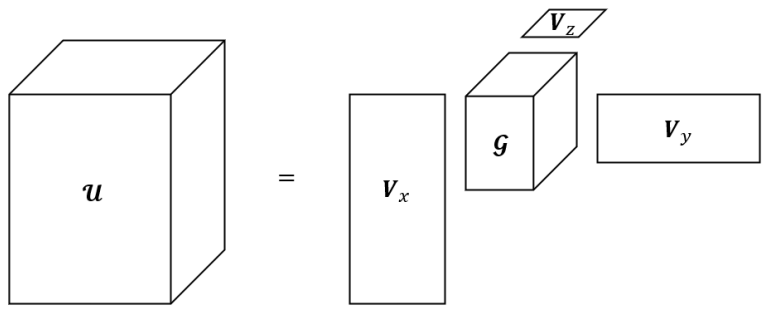
[1] L. Einkemmer, et al., (JPC., 2025)

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Tucker, hierarchical Tucker, tensor train, tree tensor networks, etc...

- **Idea:** store solution in low-rank tensor to exploit low-rank structures!
- Has seen recent popularity in the computational plasma physics community[1].

[1] L. Einkemmer, et al., (JPC., 2025)

The Model of Interest

The Vlasov-Fokker-Planck (VFP) kinetic-ion system

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla_x f_\alpha + \frac{q_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f_\alpha = \sum_{\beta=1}^{N_s} C_{\alpha\beta} \tag{2}$$

$f_\alpha(\mathbf{x}, \mathbf{v}, t) =$ particle distribution function for species α with charge q_α, mass m_α

$C_{\alpha\beta}$ is the Lenard-Bernstein-Fokker-Planck (LBFP) collision operator describing the small-angle binary collisions between species α and species β,

$$C_{\alpha\beta} = \nu_{\alpha\beta} \nabla_v \cdot \left(\frac{T_\beta}{m_\alpha} \nabla_v f_\alpha + (\mathbf{v} - \mathbf{u}_\beta) f_\alpha \right), \tag{3}$$

$\nu =$ collision frequency, $\mathbf{u} =$ drift/bulk velocity, $T =$ temperature.

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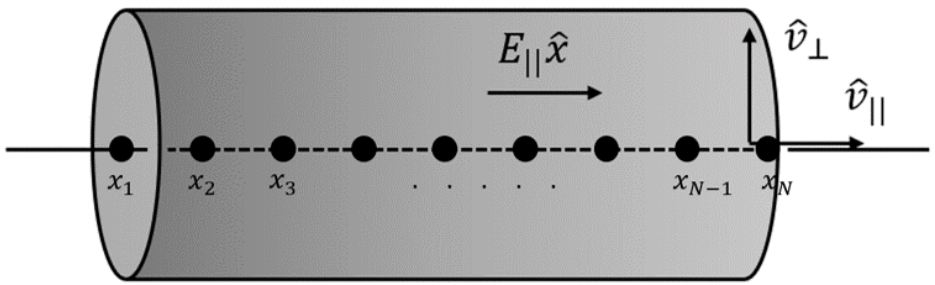
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1D-2V VFP system in cylindrical coordinates

As a first step, consider 1D-2V phase space $(x, v_\perp, v_{||})$ for a single ion species α

- $\mathbf{E} = E_{||}\hat{x}, \quad \hat{v_{||}}$ aligned with the $\hat{x}$ direction.
- $\mathbf{B} = \mathbf{0}$



1D-2V VFP system in cylindrical coordinates

$$\frac{\partial f}{\partial t} + v_{||}\frac{\partial f}{\partial x} + \frac{q_\alpha}{m_\alpha}E_{||}\frac{\partial f}{\partial v_{||}} = C_{\alpha\alpha} + C_{\alpha e}, \tag{4}$$

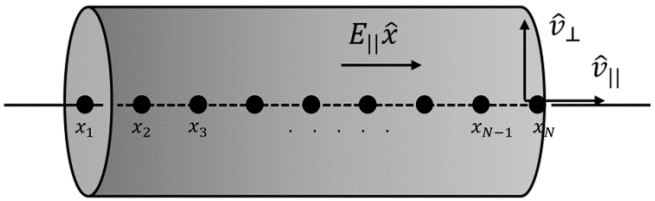
where $C_{\alpha\alpha}, C_{\alpha e}$ are defined by equation (3).

We couple the kinetic ion model with the fluid-electron energy equation[2].

$$\frac{3}{2}\frac{\partial p_e}{\partial t} + \frac{5}{2}\frac{\partial}{\partial x}(u_{||}p_e) - u_{||}\frac{\partial p_e}{\partial x} - \frac{\partial}{\partial x}\left(\kappa_{||}\frac{\partial T_e}{\partial x}\right) = -\left\langle \frac{m_\alpha |\mathbf{v}|^2}{2}, C_{\alpha e} \right\rangle_v, \tag{5}$$

[2]W. Taitano, et al., (CPC, 2021)

The kinetic-ion semi-discrete formulation



Idea:

- Discretize over $\Omega_x = [a, b]$ using spatial mesh

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \ldots < x_{N_x - \frac{1}{2}} < x_{N_x + \frac{1}{2}} = b$$

$$I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], \quad \Delta x = |I_i|$$

- Discretize over $\Omega_v = [0, R_\perp] \times [-R_{||}, R_{||}]$ using a velocity mesh $\mathbf{v}_\perp \otimes \mathbf{v}_{||}$
- At each x_i solve 2V velocity system

$$f_i(v_\perp, v_{||}, t) \approx f(x = x_i, v_\perp, v_{||}, t)$$

- Alternate between solving the kinetic-ion and fluid equations.
  - Couple the fluid-electron equation with the fluid ion moment equations

$$\underbrace{n = \langle f, 1 \rangle_v}_{\text{number density}}, \qquad \underbrace{\mathbf{u} = \frac{1}{n}\langle f, \mathbf{v} \rangle_v}_{\text{bulk velocity}}, \qquad \underbrace{T = \frac{m}{3n}\langle f, |\mathbf{v} - \mathbf{u}|^2 \rangle_v}_{\text{temperature}} \tag{6}$$

where $\langle \cdot, \cdot \rangle_v$ is the unweighted ℓ^2 inner product in velocity space.

The semi-discrete formulation

$$\frac{\partial f}{\partial t} + \underbrace{v_{||}\frac{\partial f}{\partial x}}_{\text{evolve explicitly}} + \underbrace{\frac{q_\alpha}{m_\alpha}E_{||}\frac{\partial f}{\partial v_{||}} = C_{\alpha\alpha} + C_{\alpha e}}_{\text{evolve implicitly}}, \tag{7}$$

- Discretize the PDE in physical space:

$$\frac{\partial f_i}{\partial x} \approx \max(v_{||}, 0)\frac{f_i - f_{i-1}}{\Delta x} + \min(v_{||}, 0)\frac{f_{i+1} - f_i}{\Delta x} \tag{8}$$

$$E_{||,i} = \frac{1}{q_e n_i}\frac{(nT_e)_{i+1} - (nT_e)_{i-1}}{2\Delta x}, \quad \text{from Ohm's Law} \tag{9}$$

- Discretize the matrix solution over velocity mesh: $\mathbf{f}_i \in \mathbb{R}^{N_\perp \times N_\parallel}$

  $\frac{q_\alpha E_\parallel}{m_\alpha}\frac{\partial f_i}{\partial v_{||}} \approx \mathbf{f}_i \mathbf{D}_i^T$, $\mathbf{D}_i$ is an upwind, second-order differentiation matrix.

- Discretize Fokker-Planck operators using Chang-Cooper method[3]:

$$C_{\alpha\alpha} + C_{\alpha e} \approx \nu_{\alpha\alpha,i}\left(\mathbf{C}^{\perp}_{\alpha\alpha,i} + \mathbf{C}^{||}_{\alpha\alpha,i}\right) + \nu_{\alpha e,i}\left(\mathbf{C}^{\perp}_{\alpha e,i} + \mathbf{C}^{||}_{\alpha e,i}\right). \tag{10}$$

[3] Chang, J. S. and Cooper, G, (JCP., 1970)

The semi-discrete formulation

Combine spatial and velocity discretizations:

$$\text{For each spatial node } x_i, \quad i = 1, 2 ... N_x$$

$$\frac{d\mathbf{f}_i}{dt} + \frac{1}{\Delta x}(\mathbf{f}_i - \mathbf{f}_{i-1}) * \max(\mathbf{v}_{||}, \mathbf{0})^T + \frac{1}{\Delta x}(\mathbf{f}_{i+1} - \mathbf{f}_i) * \min(\mathbf{v}_{||}, \mathbf{0})^T + \mathbf{f}_i \mathbf{D}_i^T = \mathbf{C}_i^{\perp} \mathbf{f}_i + \mathbf{f}_i \mathbf{C}_i^{||,T}, \tag{11}$$

where $*$ denotes the Hadamard/element-wise product.

Note: this is the full-rank formulation.

Rank-adaptive integration

Solutions to kinetic equations such as Fokker-Planck and Vlasov-Fokker-Planck systems exhibit low-rank structure $r_i \ll \min\{N_\perp, N_{||}\}$ in velocity space.

$$\Downarrow$$

Decompose $\mathbf{f}_i$ into its $\mathbf{V}_i^{\perp}$ and $\mathbf{V}_i^{\parallel}$ bases

$$\mathbf{f}_i(t) \approx \mathbf{V}_i^{\perp}(t)\mathbf{S}_i(t)\left(\mathbf{V}_i^{||}(t)\right)^T, \tag{12}$$

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- Update $\mathbf{V}_i^\perp, \mathbf{S}_i, \mathbf{V}_i^{||}$ using high-order *implicit-explicit* methods.
- Projection-based method.
  - K-step: Update $\mathbf{V}_i^\perp$ by projecting out $\mathbf{V}_i^{||}$ dependence.
  - L-step: Update $\mathbf{V}_i^{||}$ by projecting out $\mathbf{V}_i^\perp$ dependence.
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Storage savings: $\mathcal{O}(N_x N_v^2) \longrightarrow \mathcal{O}(N_x N_v r)$, assuming $N_v = N_\perp = N_{||}$

Rank-adaptive semi-discrete formulation

Plugging the rank-adaptive solution (12) into equation (11),

$$\frac{d}{dt}\left(\mathbf{V}_i^{\perp}\mathbf{S}_i\mathbf{V}_i^{||,T}\right)+\mathcal{F}_i+\mathbf{V}_i^{\perp}\mathbf{S}_i\left(\mathbf{D}_i\mathbf{V}_i^{||}\right)^T=\mathbf{C}_i^{\perp}\mathbf{V}_i^{\perp}\mathbf{S}_i\mathbf{V}_i^{||,T}+\mathbf{V}_i^{\perp}\mathbf{S}_i\left(\mathbf{C}_i^{||}\mathbf{V}_i^{||}\right)^T, \tag{13}$$

where

$$\begin{aligned}\mathcal{F}_i=&\frac{1}{\Delta x}\left[\mathbf{V}_i^{\perp},\mathbf{V}_{i-1}^{\perp}\right]\begin{bmatrix}\mathbf{S}_i & 0\\ 0 & -\mathbf{S}_{i-1}\end{bmatrix}\left(\left[\mathbf{V}_i^{||},\mathbf{V}_{i-1}^{||}\right]*\max(\mathbf{v}_{||},\mathbf{0})\right)^T\\ &+\frac{1}{\Delta x}\left[\mathbf{V}_{i+1}^{\perp},\mathbf{V}_i^{\perp}\right]\begin{bmatrix}\mathbf{S}_{i+1} & 0\\ 0 & -\mathbf{S}_i\end{bmatrix}\left(\left[\mathbf{V}_{i+1}^{||},\mathbf{V}_i^{||}\right]*\min(\mathbf{v}_{||},\mathbf{0})\right)^T.\end{aligned} \tag{14}$$

Rank-adaptive fully discrete formulation

We discretize equation (13) in time using a second-order implicit-explicit (IMEX) Runge-Kutta (RK) method.

For simplicity, we illustrate first-order IMEX ($\mathbf{V}_i^{\cdot,k}$ indicates the basis at time t^k):

$$\begin{aligned}\mathbf{V}_i^{\perp,k+1}\mathbf{S}_i^{k+1}\mathbf{V}_i^{||,k+1,T} &= \mathbf{V}_i^{\perp,k}\mathbf{S}_i^{k}\mathbf{V}_i^{||,k,T} \\ &- \Delta t\mathcal{F}_i^k - \Delta t\mathbf{V}_i^{\perp,k+1}\mathbf{S}_i^{k+1}\left(\mathbf{D}_i^{k+1}\mathbf{V}_i^{||,k+1}\right)^T \\ &+ \Delta t\mathbf{C}_i^{\perp,k+1}\mathbf{V}_i^{\perp,k+1}\mathbf{S}_i^{k+1}\mathbf{V}_i^{||,k+1,T} \\ &+ \Delta t\mathbf{V}_i^{\perp,k+1}\mathbf{S}_i^{k+1}\left(\mathbf{C}_i^{||,k+1}\mathbf{V}_i^{||,k+1}\right)^T,\end{aligned} \tag{15}$$

RAIL extended to cylindrical coordinates

We extend the Reduced Augmentation Implicit Low-Rank (RAIL) method[4] to solve the 2V system in cylindrical coordinates:

- Need to use the *weighted* inner product $\langle A, B \rangle_{v_\perp} := \int_0^\infty AB v_\perp dv_\perp$ to update $\mathbf{V}_i^\perp$
- We multiply and divide by $\sqrt{\mathbf{v}_\perp}$ when applying the SVD and reduced QR factorization compressions, which are numerically computed in the *unweighted* ℓ^2 inner product.
  - `wSVD` $(\mathbf{U}_w \mathbf{\Sigma} \mathbf{V}^T)$
  - `wQR` $(\mathbf{Q}_w \mathbf{R})$

[4] J. Nakao et al., (SISC, 2025)

K-Step

Goal: Update basis $\mathbf{V}_i^{\perp}$. Want to project equation (15) onto $\mathbf{V}_i^{\|,k}$ to remove the $\mathbf{v}_{\|}$-dependence.

[5] Reduced Augmentation (redaug) is a compression technique to remove redundant vectors.

K-Step

Goal: Update basis $\mathbf{V}_i^{\perp}$. Want to project equation (15) onto $\mathbf{V}_i^{\parallel,k}$ to remove the $\mathbf{v}_{\parallel}$-dependence.

$$\mathbf{K} = \mathbf{V}_i^{\perp}\mathbf{S}_i\mathbf{V}_i^{||,T}\mathbf{V}_i^{||,k} \in \mathbb{R}^{N_{\perp}\times r_i^k}. \tag{16}$$

Then, projecting equation (15) onto $\mathbf{V}_i^{\parallel,k}$, we get the Sylvester equation

$$\left(\mathbf{I}-\Delta t\mathbf{C}_i^{\perp,k+1}\right)\mathbf{K}^{k+1}+\mathbf{K}^{k+1}\left(\Delta t\left(\mathbf{D}_i^{k+1}-\mathbf{C}_i^{||,k+1}\right)\mathbf{V}_i^{||,k}\right)^T\mathbf{V}_i^{||,k} = \mathbf{K}^k-\Delta t\mathcal{F}_i^k\mathbf{V}_i^{||,k}. \tag{17}$$

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The orthonormal basis $\mathbf{V}_i^{\perp}$ can be extracted via

$$\tilde{\mathbf{V}}_i^{\perp,k+1}\mathbf{R} := \mathbf{Q}_w\mathbf{R} = \texttt{wQR}(\mathbf{K}_i^{k+1}). \tag{18}$$

(weighted QR for *weighted* ℓ^2 inner product space)

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(weighted QR for *weighted* ℓ^2 inner product space)
Augment[5] with the previous bases:

$$\hat{\mathbf{V}}_i^{\perp,k+1} = \texttt{redaug}\left[\tilde{\mathbf{V}}_i^{\perp,k+1}, \mathbf{V}_i^{\perp,k}\right]. \tag{19}$$

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L-Step

Goal: Update basis $\mathbf{V}_i^{||}$. Want to project equation (15) onto $\mathbf{V}_i^{\perp,k}$ to remove the $v_\perp$-dependence.

$$\mathbf{L} = \mathbf{V}_i^{||}\mathbf{S}_i^T\mathbf{V}_i^{\perp,T}\big(\mathbf{v}_\perp * \mathbf{V}_i^{\perp,k}\big). \tag{20}$$

Then, projecting equation (15) onto $\mathbf{V}_i^{\perp,k}$ we get the Sylvester equation

$$\Big(\mathbf{I} + \Delta t\big(\mathbf{D}_i^{k+1} - \mathbf{C}_i^{||,k+1}\big)\Big)\mathbf{L}^{k+1} + \mathbf{L}^{k+1}\Big(-\Delta t\mathbf{C}_i^{\perp,k+1}\mathbf{V}_i^{\perp,k}\Big)^T\big(\mathbf{v}_\perp * \mathbf{V}_i^{\perp,k}\big) = \mathbf{L}^k - \Delta t\mathcal{F}_i^{k,T}\big(\mathbf{v}_\perp * \mathbf{V}_i^{\perp,k}\big). \tag{21}$$

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$$\tilde{\mathbf{V}}_i^{||,k+1}\mathbf{R} := \mathbf{Q}\mathbf{R} = \mathtt{QR}(\mathbf{L}_i^{k+1}) \tag{22}$$

(standard QR for *unweighted* ℓ^2 inner product space)
Augment with the previous bases:

$$\hat{\mathbf{V}}_i^{||,k+1} = \mathtt{redaug}\Big[\tilde{\mathbf{V}}_i^{||,k+1}, \mathbf{V}_i^{||,k}\Big] \tag{23}$$

S-Step

Goal: update $\mathbf{S}_i$ using Galerkin projection.

$$\hat{\mathbf{S}}_i = \left(\mathbf{v}_\perp * \hat{\mathbf{V}}_i^{\perp,k+1}\right)^T \mathbf{V}_i^{\perp} \mathbf{S}_i \mathbf{V}_i^{||,T} \hat{\mathbf{V}}_i^{||,k+1} \tag{24}$$

Project the equation and solve the Sylvester equation

$$\begin{aligned} &\left(\mathbf{I} - \Delta t \left(\mathbf{v}_\perp * \hat{\mathbf{V}}_i^{\perp,k+1}\right)^T \left(\mathbf{C}_i^{\perp,k+1} \hat{\mathbf{V}}_i^{\perp,k+1}\right)\right) \hat{\mathbf{S}}_i^{k+1} \\ &\quad + \hat{\mathbf{S}}_i^{k+1} \left(\Delta t \left(\left(\mathbf{D}_i^{k+1} - \mathbf{C}_i^{||,k+1}\right) \hat{\mathbf{V}}_i^{||,k+1}\right)^T\right) \hat{\mathbf{V}}_i^{||,k+1} \\ &\quad = \hat{\mathbf{S}}_i^k - \Delta t \left(\mathbf{v}_\perp * \hat{\mathbf{V}}_i^{\perp,k+1}\right)^T \mathcal{F}_i^k \hat{\mathbf{V}}_i^{||,k+1}. \end{aligned} \tag{25}$$

Output: $\hat{\mathbf{S}}_i$

Need to truncate $\hat{\mathbf{V}}_i^{\perp}, \hat{\mathbf{S}}_i, \hat{\mathbf{V}}_i^{\|}$ to small tolerance ϑ to maintain low-rank structure.

LoMaC truncation in azimuthally symmetric cylindrical coordinates

Problem: SVD truncation only conserves moments up to the tolerance ϑ.
Fix: Local Macroscopic Conservative (LoMaC) truncation[6].

[6] W. Guo and J.-M. Qiu, (JSC, 2024)

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1. Scale pre-truncated solution $\mathbf{f}^\star$ by a Gaussian weight: $\tilde{\mathbf{f}} \coloneqq \mathbf{f}^\star / \mathbf{w}, \quad \mathbf{w} = \mathbf{w}_\perp \otimes \mathbf{w}_{||}$.

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2. Project using $\langle \cdot, \cdot \rangle_{\mathbf{w}}$ onto the subspace

$$\mathcal{N} \coloneqq \text{span}\{\mathbf{1},\ \mathbf{v}_{||},\ \mathbf{v}_\perp^2 + \mathbf{v}_{||}^2\}.$$

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3. Truncate the orthogonal complement:

$$\mathbf{f} = \mathbf{f}_1^M + (I - P)\big(\mathcal{T}_\vartheta(\mathbf{f}_2)\big), \qquad \mathbf{f}_2 = (I - P)(\mathbf{f}^\star). \tag{26}$$

$\mathcal{T}_\vartheta :=$ `wQR-SVD` truncation procedure to tolerance ϑ.

[6] W. Guo and J.-M. Qiu, (JSC, 2024)

The fluid solver

Need: macroscopic moments $n_\alpha, u_{\parallel,\alpha}, T_\alpha, T_e$ at future RK stages to construct the discrete $E_\parallel$ and collision operators.

$$\frac{dn_i}{dt} = -\frac{1}{\Delta x}\left[(\widehat{nu_{||}})_{i+\frac{1}{2}} - (\widehat{nu_{||}})_{i-\frac{1}{2}}\right], \tag{27}$$

$$\frac{d(nu_{||})_i}{dt} = -\frac{1}{\Delta x}\left[\hat{S}_{i+\frac{1}{2}} - \hat{S}_{i-\frac{1}{2}}\right] + \frac{q_\alpha}{2\Delta x m_\alpha q_e}\left[(nT_e)_{i+1} - (nT_e)_{i-1}\right], \tag{28}$$

$$\begin{aligned}\frac{d(nU)_i}{dt} = &-\frac{1}{\Delta x}\left[\hat{Q}_{i+\frac{1}{2}} - \hat{Q}_{i-\frac{1}{2}}\right] + \frac{q_\alpha u_{||,i}}{2\Delta x m_\alpha q_e}\left[(nT_e)_{i+1} - (nT_e)_{i-1}\right] \\ &+ \frac{3\sqrt{2m_e}}{m_\alpha^2 T_{e,i}^{3/2}}\left[n_i^2 T_{e,i} - \frac{m_\alpha}{3}\left(2n_i(nU)_i - (nu_{||})_i^2\right)\right],\end{aligned} \tag{29}$$

where the numerical fluxes are defined as

$$\begin{pmatrix}\widehat{nu_{||}} \\ \hat{S} \\ \hat{Q}\end{pmatrix}_{i+\frac{1}{2}} = 2\pi\int_{-\infty}^{\infty}\int_{0}^{\infty} v_{||}\hat{f}_{i+\frac{1}{2}}\begin{pmatrix}1 \\ v_{||} \\ (v_\perp^2 + v_{||}^2)/2\end{pmatrix} v_\perp dv_\perp dv_{||}, \tag{30}$$

The fluid solver

Discretizing the electron energy equation (5) in x,

$$\begin{aligned}\frac{d(nT_e)_i}{dt} =& -\frac{5}{3\Delta x}\left[\hat{u}_{||,i+\frac{1}{2}}\widehat{(nT_e)}_{i+\frac{1}{2}} - \hat{u}_{||,i-\frac{1}{2}}\widehat{(nT_e)}_{i-\frac{1}{2}}\right] + \frac{(nu_{||})_i}{3n_i\Delta x}\left[(nT_e)_{i+1} - (nT_e)_{i-1}\right] \\ &+ \frac{3.2}{3\sqrt{2m_e}\Delta x^2}\left[(T_{e,i}^{5/2} + T_{e,i+1}^{5/2})(T_{e,i+1} - T_{e,i}) - (T_{e,i-1}^{5/2} + T_{e,i}^{5/2})(T_{e,i} - T_{e,i-1})\right] \\ &+ \frac{2\sqrt{2m_e}}{m_\alpha T_{e,i}^{3/2}}\left[\frac{m_\alpha}{3}\left(2n_i(nU)_i - (nu_{||})_i^2\right) - n_i^2 T_{e,i}\right],\end{aligned} \tag{31}$$

where

$$\hat{u}_{||,i+\frac{1}{2}} = \frac{u_{||,i} + u_{||,i+1}}{2}, \qquad \widehat{(nT_e)}_{i+\frac{1}{2}} = \begin{cases} n_i T_{e,i}, & \hat{u}_{||,i+\frac{1}{2}} > 0, \\ n_{i+1} T_{e,i+1}, & \hat{u}_{||,i+\frac{1}{2}} \leq 0. \end{cases} \tag{32}$$

Integrate with same IMEX method; use Newton solver to compute moments.

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2 The Algorithm

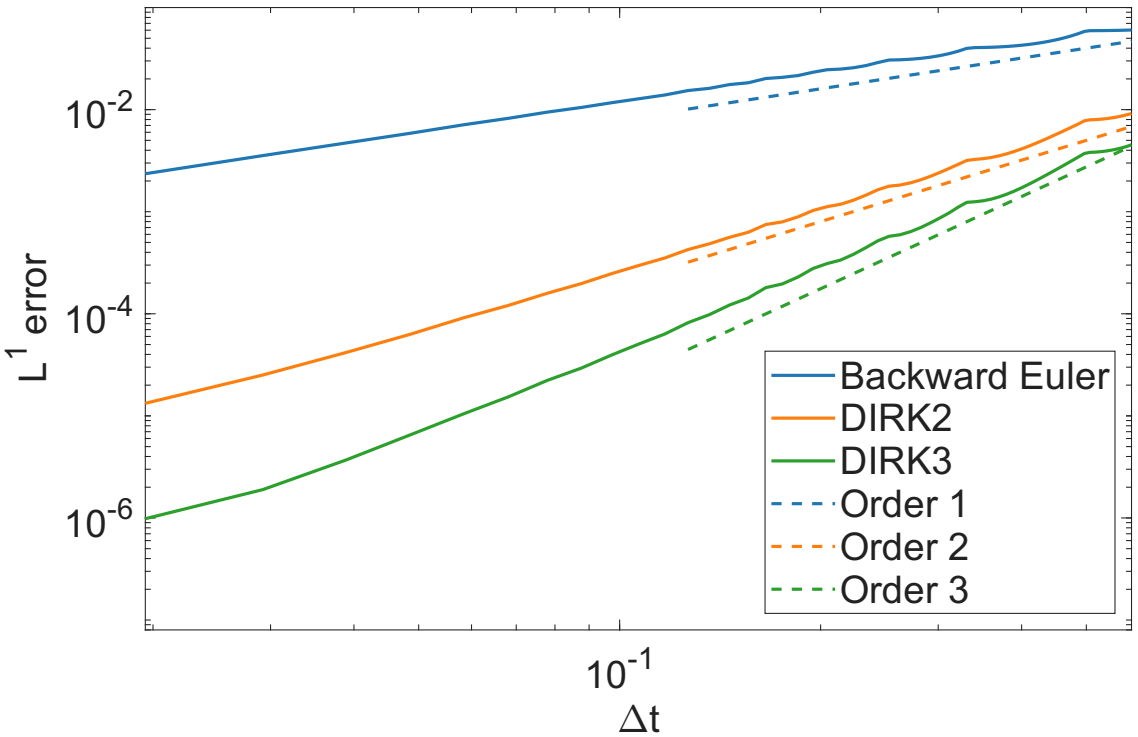
3 Numerical Results

4 Closing Remarks

Lenard-Bernstein-Fokker-Planck equation (0D-2V)

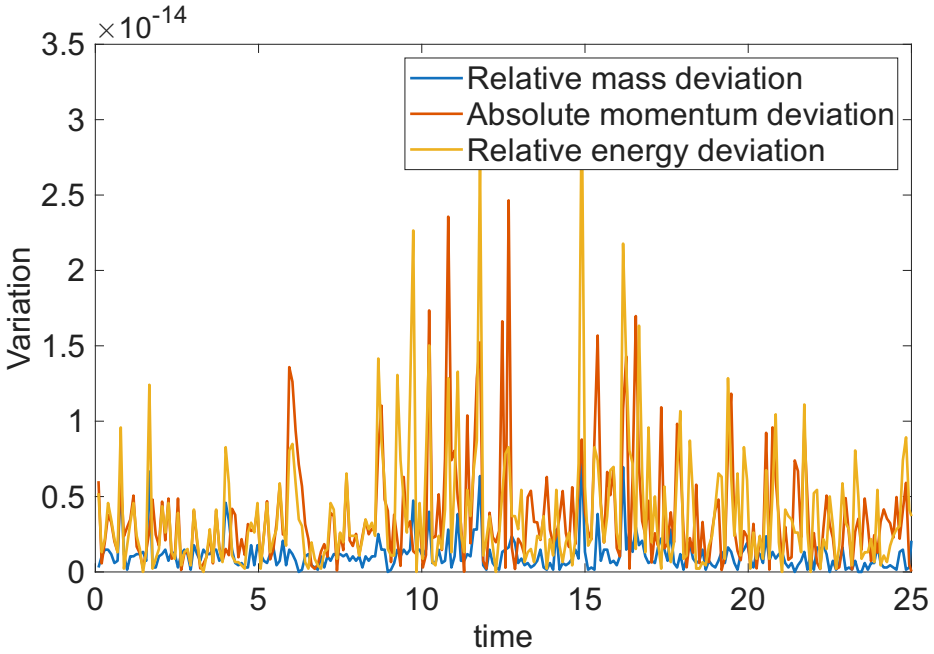
$$\frac{\partial f}{\partial t} = \nu_{\alpha\alpha} \nabla_v \cdot \left(\frac{T_\alpha}{m_\alpha} \nabla_v f + (\mathbf{v} - \mathbf{u}_\alpha) f \right), \quad (v_\perp, v_{||}) \in (0, 14) \times (-16, 16)$$

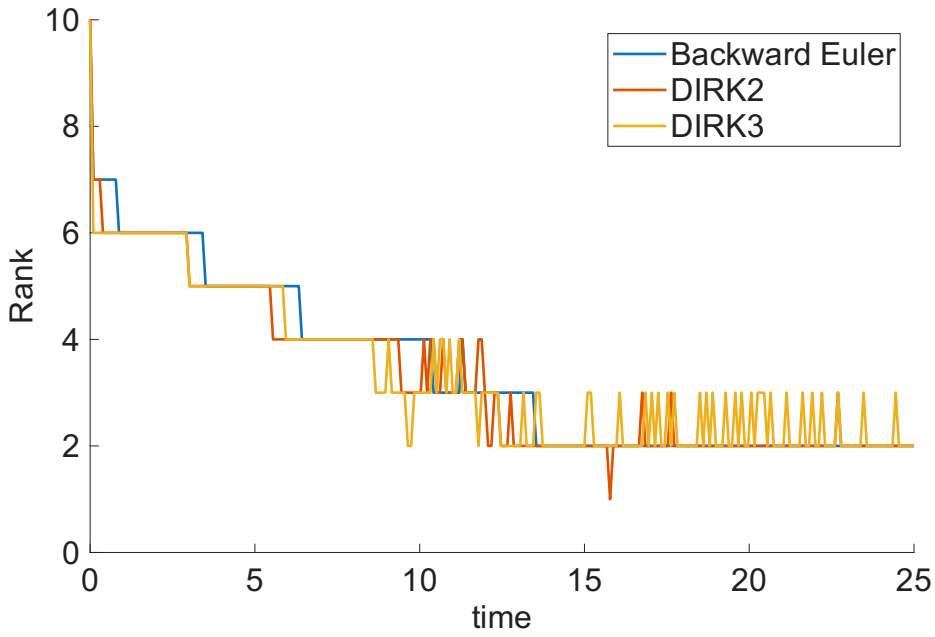
IC: Rank-2 Maxwellian $f(t = 0) = f_{M_1} + f_{M_2}$
Equilibrium: Maxwellian f_∞ with moments of the IC.



Temporal error, backward Euler / DIRK2 / DIRK3. $N_\perp = N_{||} = 100$, $T_f = 1$, $\vartheta = 10^{-6}$; reference: DIRK3, $\Delta t = 0.00974$.

LBFP equation: conservation and rank

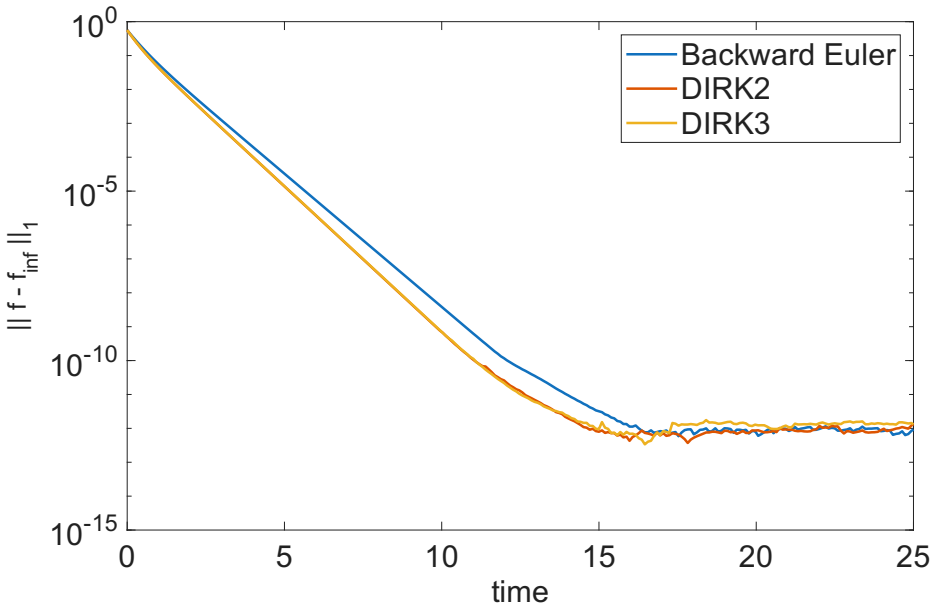


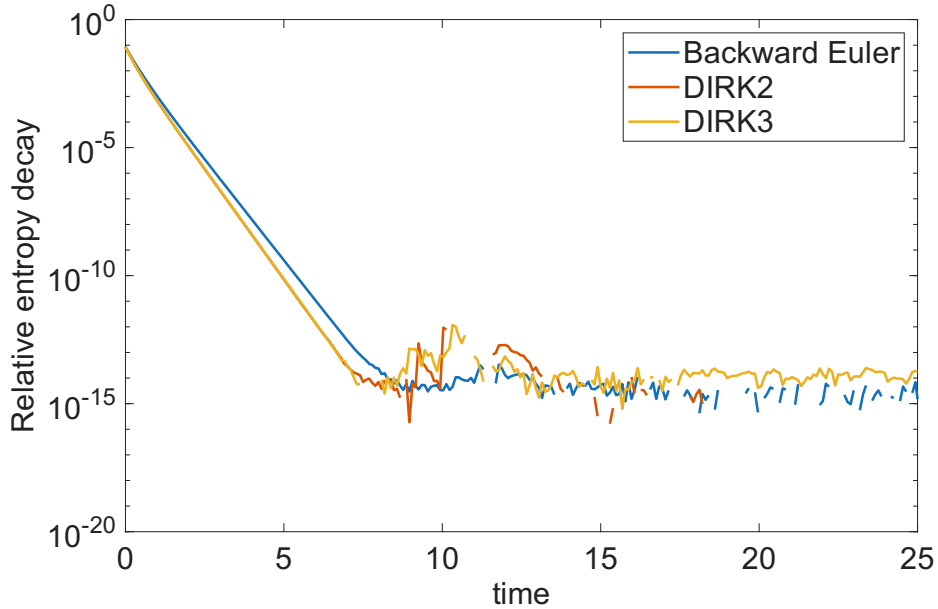


Relative change in mass, momentum, energy with DIRK3 *(left)*; solution rank *(right)*. $N_\perp = N_{||} = 100$, $\vartheta = 10^{-6}$, $\Delta t = 1/(1/\Delta_\perp + 1/\Delta_{||})$, $T_f = 25$.

- Mass, momentum, and energy conserved to roughly machine precision.
- Rank immediately increases from rank-2 initial condition as the two Maxwellians diffuse, then relaxes to the low-rank equilibrium.

LBFP equation: equilibrium preservation and entropy





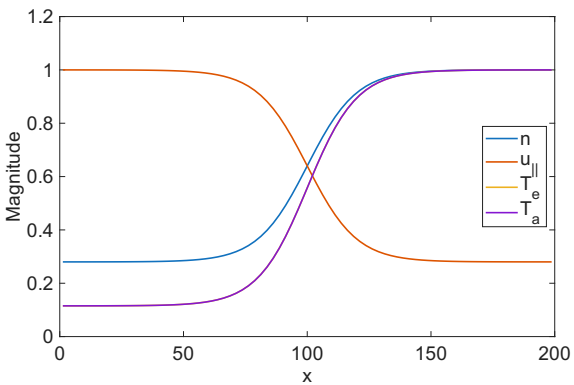
L^1 decay $\|f - f_\infty\|_1$ *(left)*; Kullback relative entropy $\int f \log(f/f_\infty)\, v_\perp\, d^3v$ *(right)*. Same parameters as previous slide.

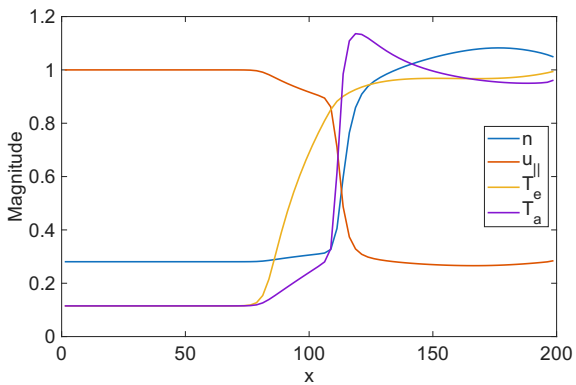
- Equilibrium preservation and entropy dissipation to roughly machine precision, despite the larger tolerance $\vartheta = 10^{-6}$
  $\Rightarrow$ LoMaC projects onto a subspace that captures the Maxwellian steady state well.

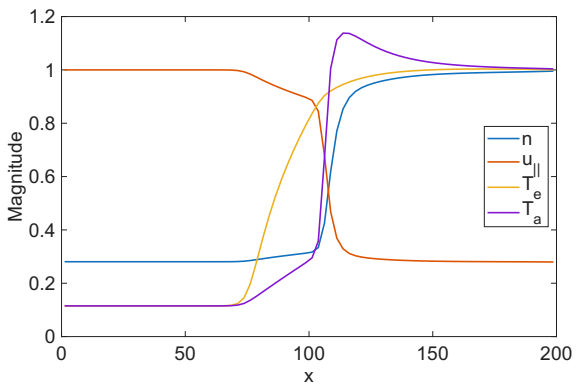
Mach 5 standing shock (1D-2V hybrid VFP)

- IC: local Maxwellian from macroscopic moment profiles, forming smoothed shock

$$n(x,0) = 0.36\tanh\left(0.05(x-100)\right) + 0.64,$$
$$u_{||}(x,0) = -1.1154\tanh\left(0.05(x-100)\right) + 1.983,$$
$$T_\alpha(x,0) = 0.4424\tanh\left(0.05(x-100)\right) + 0.5576.$$

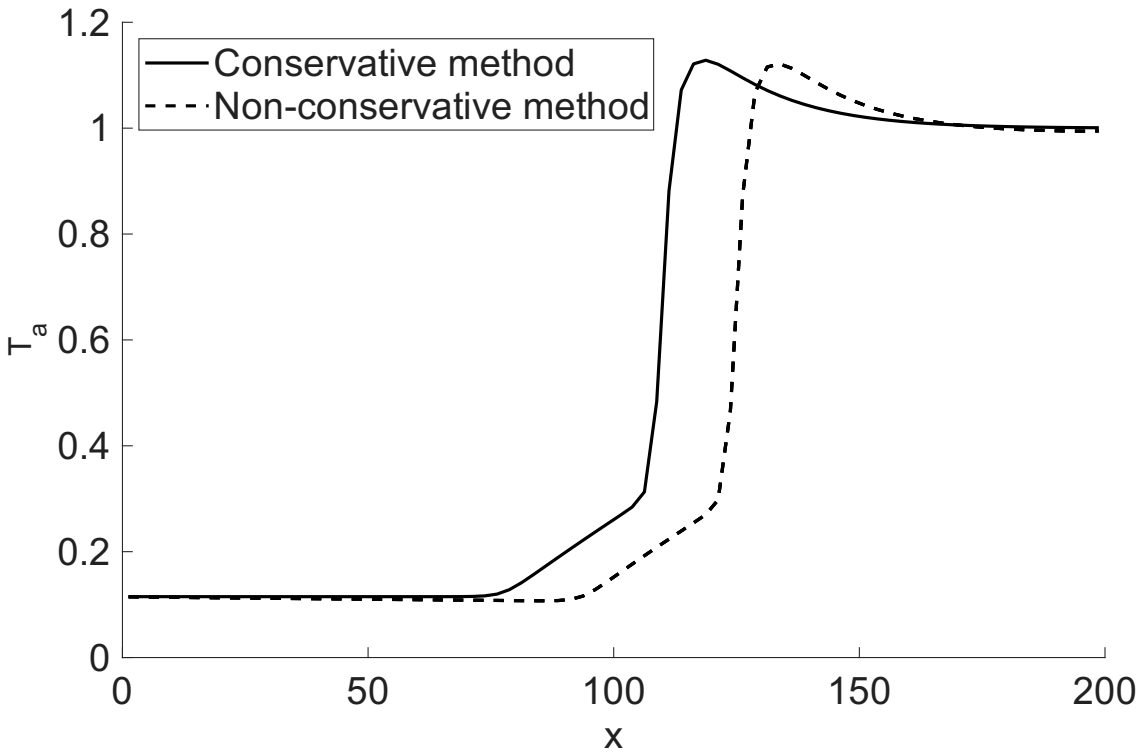






Macroscopic profiles at $t = 0, 100, 250$. $N_x = 80$, $N_\perp = N_{||} = 100$, $\vartheta = 10^{-8}$, $\Delta t = 0.4$; $u_{||}$ normalized to upstream value.

Mach 5 standing shock: testing stationary shock



Ion temperature T_α at $t = 5000$ with LoMaC *(solid)* vs. without LoMaC *(dashed)*. IMEX222, $N_x = 80$, $N_\perp = N_{||} = 100$, $\Delta t = 0.4$, $\vartheta = 10^{-8}$.

- Without conservative truncation, the ion temperature profile continues to drift nonphysically to the right as t increases.
- Conservation is critical for long-time kinetic simulations.

Mach 5 standing shock: velocity-space accuracy

Fix $N_v = 200$ in one velocity variable and refine the other ($N_v = 48, 80, 144, 240$). Reference: $N_v = 720$ in the refined variable. $N_x = 160$, $\Delta t = 0.2$, $\vartheta = 10^{-8}$, $T_f = 1$.

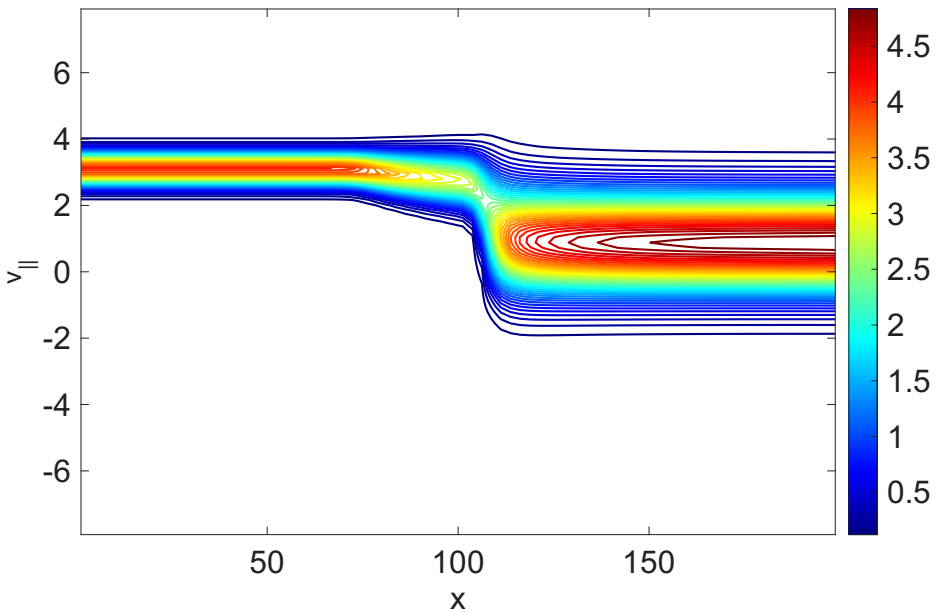
Refinement in $v_\perp$

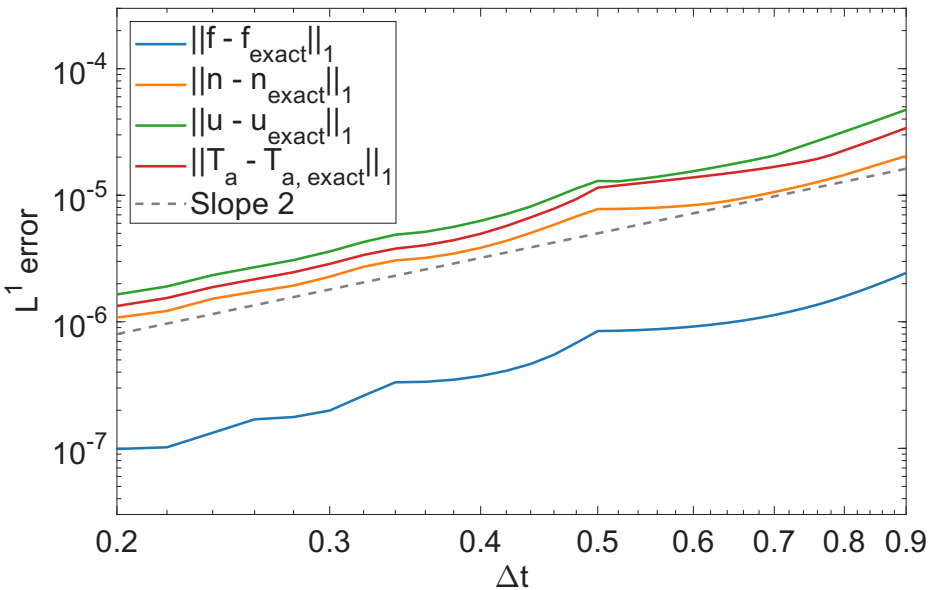
$N_\perp \times N_{\|}$	L^1	Order	L^∞	Order
48×200	1.25E-06	–	1.71E-04	–
80×200	4.10E-07	2.18	5.32E-05	2.29
144×200	1.18E-07	2.13	1.46E-05	2.20
240×200	3.80E-08	2.19	4.60E-06	2.25

Refinement in $v_{\|}$

$N_\perp \times N_{\|}$	L^1	Order	L^∞	Order
200×48	1.69E-05	–	8.90E-03	–
200×80	5.98E-06	2.04	3.20E-03	1.98
200×144	1.79E-06	2.06	1.00E-03	2.04
200×240	5.90E-07	2.15	3.00E-04	2.15

Mach 5 standing shock: distribution function and temporal accuracy

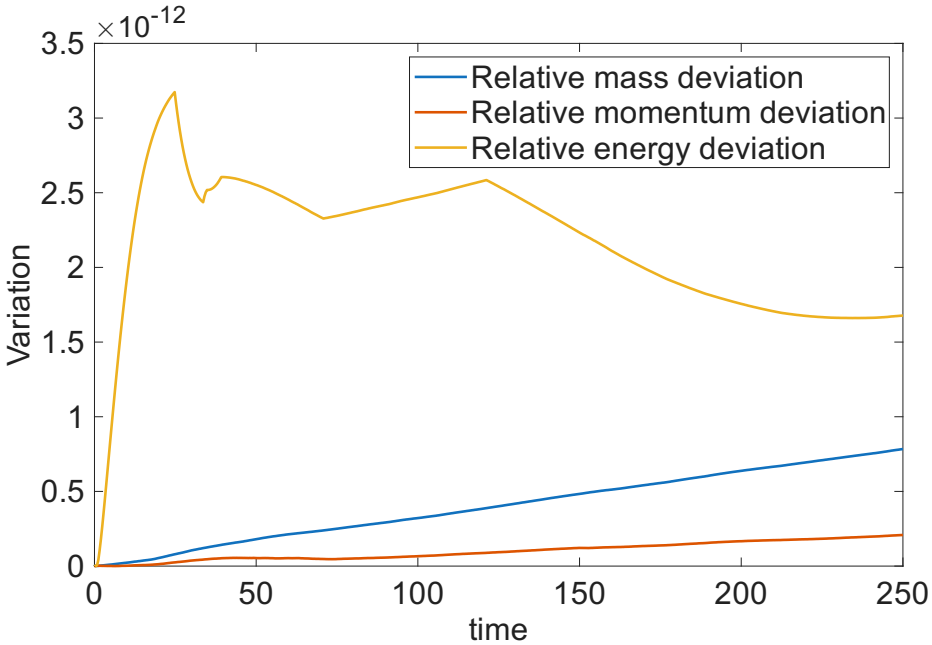


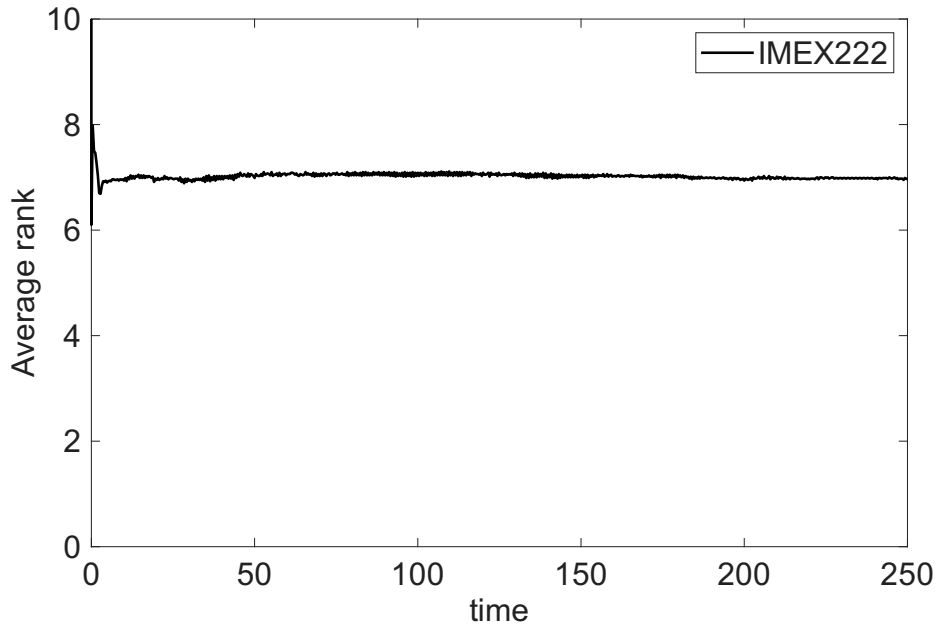


Left: $2\pi \int_{\mathbb{R}_+} f(x, v_\perp, v_{||}, t = 250)\, v_\perp \, dv_\perp$; $N_x = 80$, $N_\perp = N_{||} = 100$, IMEX222, $\Delta t = 0.4$.

Right: L^1 errors in f, n, $u_{||}$, T_α for IMEX222; $N_x = 160$, $N_\perp = N_{||} = 200$, $T_f = 1$, $\vartheta = 10^{-8}$, reference $\Delta t = 0.01$.

Mach 5 standing shock: conservation and rank





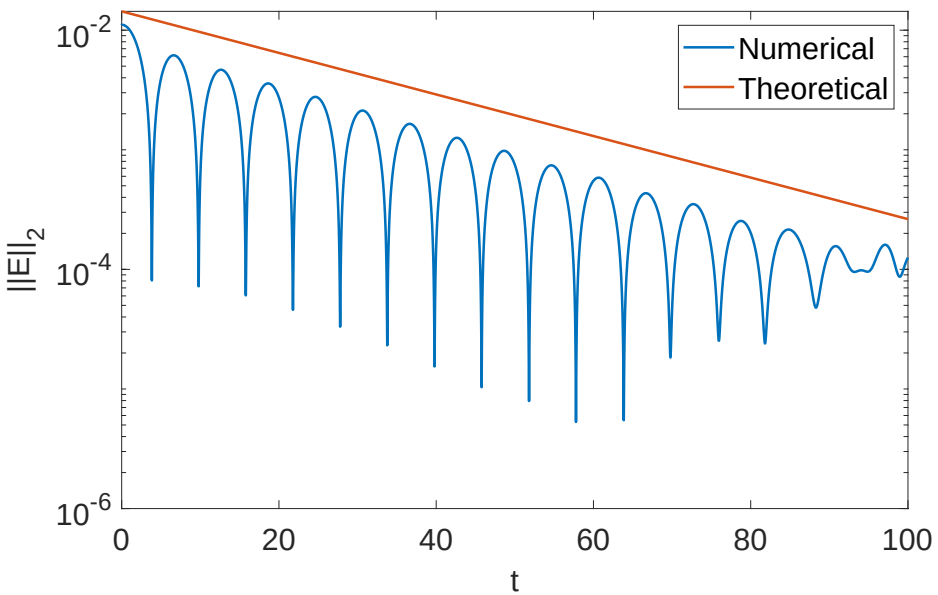
Relative change in mass, momentum, energy *(left)*; average rank over all x_i *(right)*. $N_x = 80$, $N_\perp = N_{||} = 100$, $\Delta t = 0.4$, $T_f = 250$, $\vartheta = 10^{-8}$. Dirichlet boundary contributions subtracted when computing the relative changes.

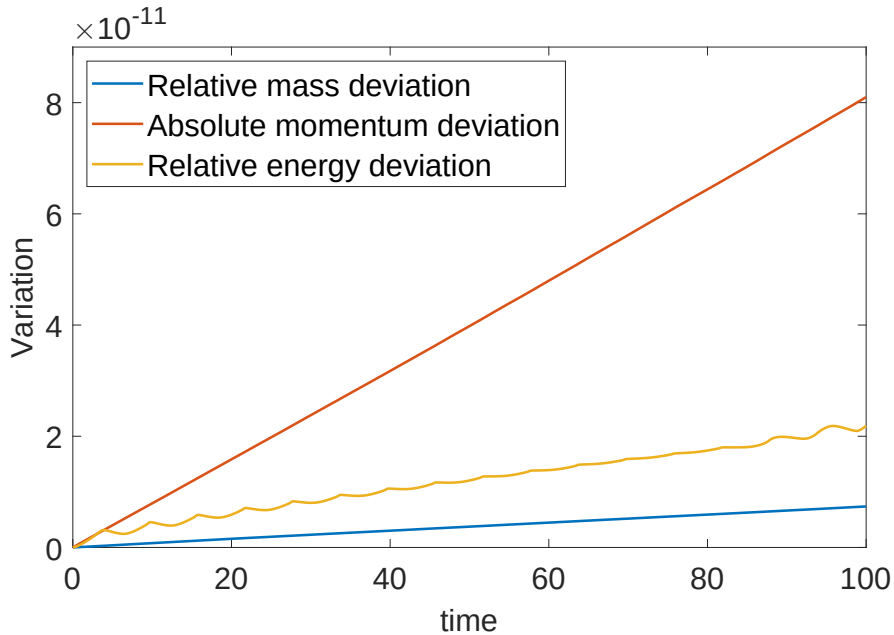
Weak Landau damping

- IC:

$$T_\alpha(x,0) = 0.2, \quad T_e(x,0) = 1, \quad u_{||} = 0, \quad n(x,0) = 1 + 0.01\sin(2\pi x/16).$$

- Set $C_{\alpha\alpha} = C_{\alpha e} = 0$, $W_{e\alpha} = 0$; scale thermal conductivity $\kappa_{||}$ by 10 so T_e stays nearly uniform (isothermal-electron limit).





$\|E_{||}(x,t)\|_2$ *(left)*; relative change in mass, momentum, energy *(right)*. $N_x = 300$, $N_\perp = 64$, $N_{||} = 128$, $\Delta t = 0.005$, $T_f = 100$, $\vartheta = 10^{-8}$.

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Concluding remarks

Recap:

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- 2nd order accuracy in time and velocity.
- Macroscopic moments well-conserved.
- Physics preserved:
  - LBFP: correct drive to equilibrium and entropy dissipation to machine precision.
  - Mach 5 standing shock: standing shock preserved over long times using LoMaC.
  - Weak Landau damping: good agreement with theoretical damping rate $\gamma_{LD} = -0.04$.

Ongoing work:

- Extend to multi-species.
- Extend to higher dimensions.

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Thank you