A Robust, Third-Order Accurate Solver for the 1D1V Vlasov-Poisson Equation

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SWARTHMORE

The model

Consider the Vlasov-Poisson advection equation

$$f_t + vf_x + E(x,t)f_v = 0 \quad (1)$$

$f(x,v,t)$ = probability density function
(x,v) = coordinates in phase-space
$E(x,t)$ = electric field

Discretization

Mesh: $I_{i,j} := [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [v_{j-\frac{1}{2}}, v_{j+\frac{1}{2}}]$
Numerical solution $f_{i,j}^n \approx f(x_i, v_j, t^n)$
Need: $E_i^n \approx E(x_i, t^n)$ and $\hat{f}_{i+\frac{1}{2},j}^n \approx f(x + \frac{\Delta x_i}{2}, v_j, t^n)$
Goal: $\{f_{i,j}^n\} \to \{f_{i,j}^{n+1}\}$

Poisson's Equation: compute $E(x,t)$

Solve Poisson's equation w/ Fourier Transform $\mathcal{F}$.

$$\phi_{xx} = -\rho(x,t) = -\left(\int_V f(x,v,t)dV - 1\right) \quad (2a)$$

$$\mathcal{F}\{-\rho\} = \mathcal{F}\{\phi_{xx}\} = ik\mathcal{F}\{\phi_x\} \quad (2b)$$

$$E = -\phi_x \Rightarrow E(x,t) = \mathcal{F}^{-1}\left\{\frac{-i}{k}\mathcal{F}\{\rho\}\right\} \quad (2c)$$

WENO5 [2]

Approximate/reconstruct cell boundary values [2].

In x, v: $\{f_{i-2}, f_{i-1}, f_i, f_{i+1}, f_{i+2}\} \to \hat{f}_{i\pm\frac{1}{2}}$

→ High Order $\mathcal{O}(\Delta x^5)$
→ Controls oscillations → resolves sharp gradients

Strong Stability-Preserving Runge-Kutta [1]

For each Runge-Kutta Stage at time $t^{(k)}$:

Step 1. Compute $E(x,t)$ via Poisson's equation.

Step 2. Compute flux differences $F_x^{(k-1)}$, $F_v^{(k-1)}$ for each cell center $f_{i,j}$

$$F_x(x,v,t) = vf, \quad F_v(x,v,t) = E(x,t)f$$

Step 3. Reconstruct cell fluxes using WENO5.

Step 4. Compute $f^{(k)}$ via SSP-RK3 scheme.

SSP-RK3 butcher tableau

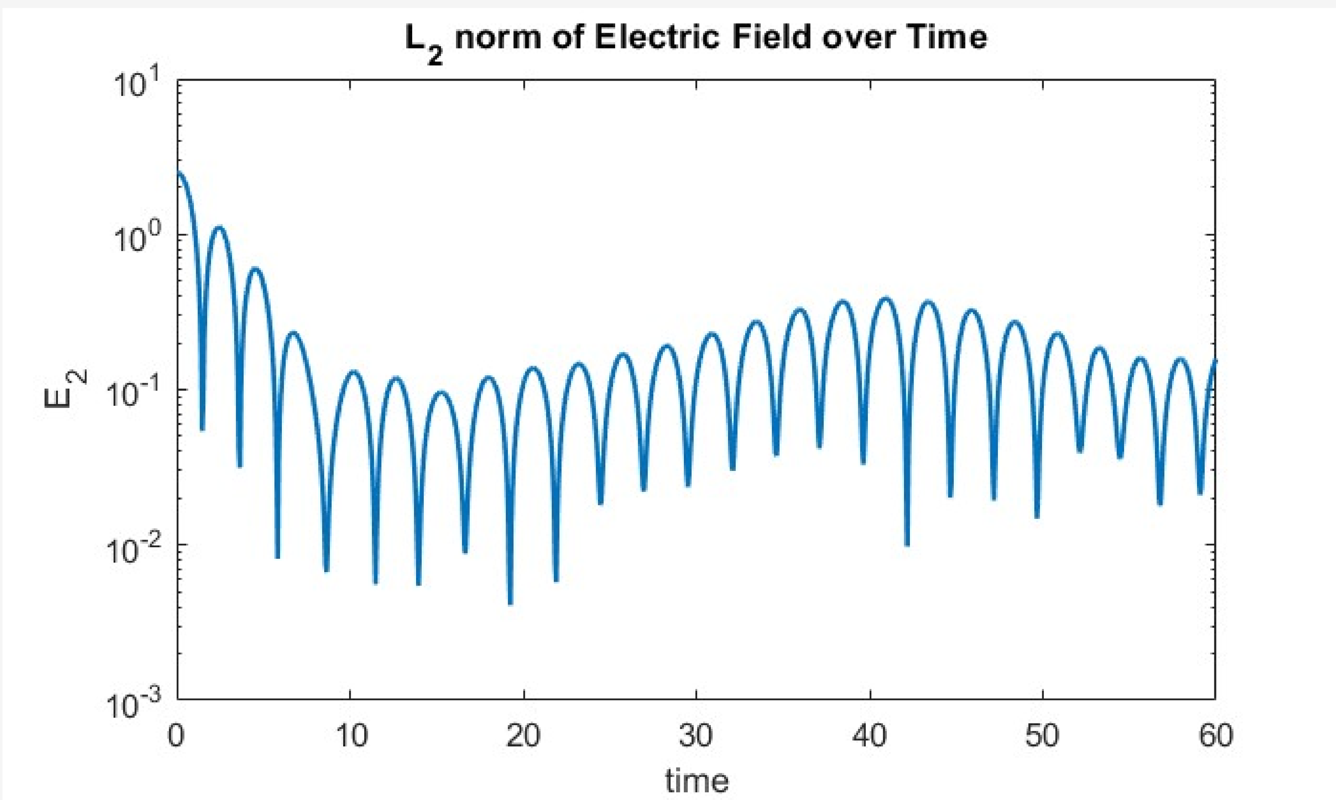
0	0	0	0
1	1	0	0
1/2	1/4	1/4	0
	1/6	1/6	2/3

$$f^{(k)} = f^{(n)} + \sum_{\ell=1}^{3} a_{k,\ell}\left(\frac{\Delta t}{\Delta x}F_x^{(\ell-1)} + \frac{\Delta t}{\Delta v}F_v^{(\ell-1)}\right) \quad (3a)$$

$$f^{(n+1)} = f^{(n)} + \sum_{\ell=1}^{3} b_{\ell}\left(\frac{\Delta t}{\Delta x}F_v^{(\ell-1)} + \frac{\Delta t}{\Delta v}F_x^{(\ell-1)}\right) \quad (3b)$$

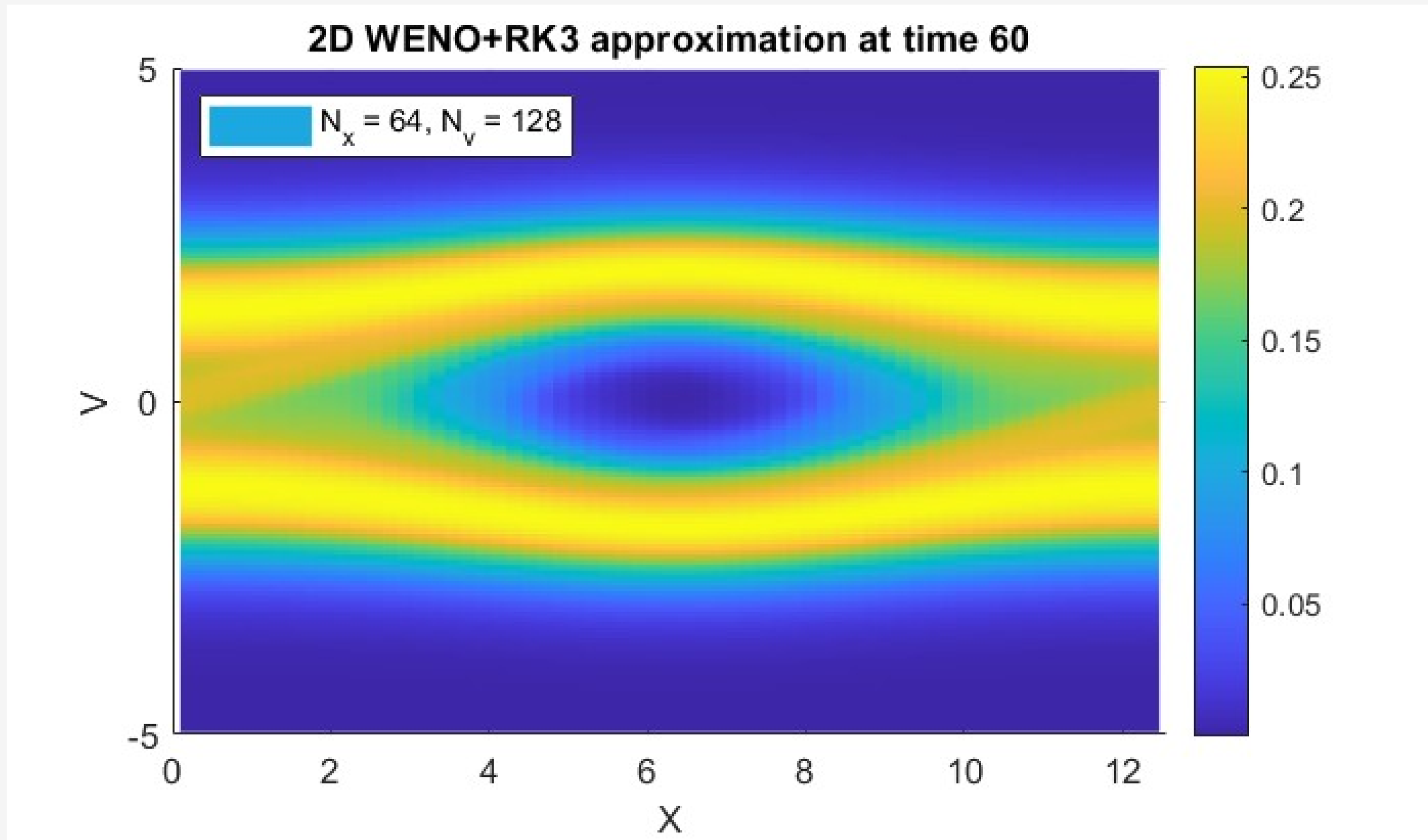
Numerics: Strong Landau Damping

$$f_0 = \frac{1}{\sqrt{2\pi}}\left(1 + \alpha\cos(kx)e^{-0.5v^2}\right), \quad \alpha = k = 0.5$$



Numerics: Two-Stream Instability

$$f_0 = \frac{1}{\sqrt{2\pi}}\left(1 + 0.05\cos(0.5x)e^{-0.5v^2}\right)v^2$$



Ongoing work: collision operators

Vlasov-Fokker-Planck model: we are currently working on low-rank, high-order implicit-explicit (IMEX) methods to compute the model

$$f_t + vf_x + E(x,t)f_v = \underbrace{C(f,f)}_{\text{collision operator}} \quad (4)$$

$C(f,f)$ requires implicit-explicit (IMEX) scheme
Working on extending to collision operators in cylindrical coordinates.

References

[1] Gottlieb, S., Shu, C.-W., & Tadmor, E. (2001). Strong Stability-Preserving High-Order Time Discretization Methods. *SIAM Review*, 43(1), 89–112.
[2] Jing-Mei Qiu, Andrew Christlieb, A conservative high order semi-Lagrangian WENO method for the Vlasov equation, *Journal of Computational Physics*, Volume 229, Issue 4, 2010, Pages 1130-1149, ISSN 0021-9991,
[3] Shu, C. (2009). High Order Weighted Essentially Nonoscillatory Schemes for Convection Dominated Problems. *SIAM Review.*, 51, 82-126.