An Implicit, Low-Rank Solver for 1D2V Vlasov-Fokker-Planck Equation in Cylindrical Coordinates

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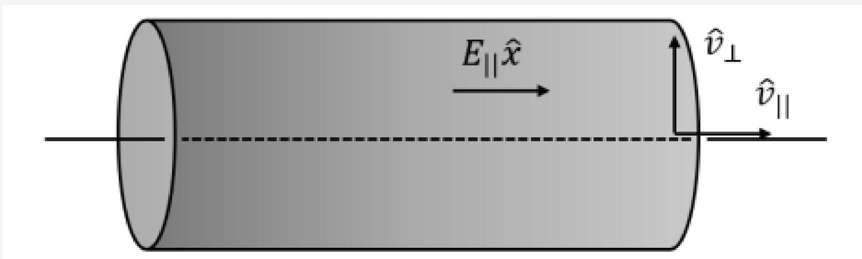
1D2V Vlasov-Fokker-Planck Model

Laboratory plasma experiments are expensive, necessitating simulations such as the Vlasov-Fokker-Planck model:

$$\frac{\partial f}{\partial t} + v_\parallel \frac{\partial f}{\partial x} + E_\parallel \frac{\partial f}{\partial v_\parallel} = C_{\alpha\alpha}(f) + C_{\alpha e}(f) \qquad (1)$$



$f(x, v_\perp, v_\parallel, t)$ = Ion probability distribution function
$E_\parallel$ = electric field in parallel (x) dimension
Coupled with fluid electron pressure equation.

Discretization: 1st order IMEX RK

$f_i^k \approx f(x_i, v_\perp, v_\parallel, t^k)$
Spatial mesh: Cells $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$
Velocity mesh: Cells $[v_{\perp,j-\frac{1}{2}}, v_{\perp,j+\frac{1}{2}}] \times [v_{\parallel,l-\frac{1}{2}}, v_{\parallel,l+\frac{1}{2}}]$

$$f_i^{k+1} + \Delta t\left(E_\parallel^{k+1}\frac{\partial f_i^{k+1}}{\partial v_\parallel} - C_{\alpha\alpha}^{k+1} - C_{\alpha e}^{k+1}\right) = f_i^k - \frac{\Delta t}{\Delta x} v_\parallel \left(\hat{f}_{i+\frac{1}{2}}^k - \hat{f}_{i-\frac{1}{2}}^k\right)$$

The Algorithm

Naively, storage complexity is $\mathcal{O}(N_x N_v^2)$
Assume low-rank structure ($r \ll N_v$): $\mathbf{f}_i^k = \mathbf{V}_\perp^k \mathbf{S}^k (\mathbf{V}_\parallel^k)^T$
Storage reduction: $\mathcal{O}(N_x(r^2 + 2N_v r))$

At each timestep t^k:

- Compute moments at next timestep n^{k+1}, $(nu_\parallel)^{k+1}$, T_α^{k+1}, T_e^{k+1} using Newton method
- **At each spatial node x_i:**
  - Compute numerical fluxes $\hat{f}_{i\pm\frac{1}{2}}^k$
  - Compute electric field: $E_\parallel^{k+1} = \frac{1}{q_e n_e^{k+1}} \frac{(n_e T_e)_{i+1}^{k+1} - (n_e T_e)_{i-1}^{k+1}}{2\Delta x}$
  - Compute collision operators $C_{\alpha\alpha}^{k+1}, C_{\alpha e}^{k+1}$
  - Solve for f_i^{k+1} using low-rank projection [1].

$$\left.\begin{aligned} \mathbf{K}^k &= \mathbf{V}_\perp^k \mathbf{S}^k (\mathbf{V}_\parallel^k)^T \mathbf{V}_{\parallel,\star}^{k+1} \\ \mathbf{L}^k &= \mathbf{V}_\parallel^k (\mathbf{S}^k)^T (\mathbf{V}_\perp^k)^T \mathbf{V}_{\perp,\star}^{k+1} \end{aligned}\right\} \text{ Basis update}$$

$$\left.\mathbf{S}^k = (\mathbf{V}_{\perp,\star}^{k+1})^T \mathbf{V}_\perp^k \mathbf{S}^k (\mathbf{V}_\parallel^k)^T \mathbf{V}_{\parallel,\star}^{k+1}\right\} \text{ Galerkin projection}$$

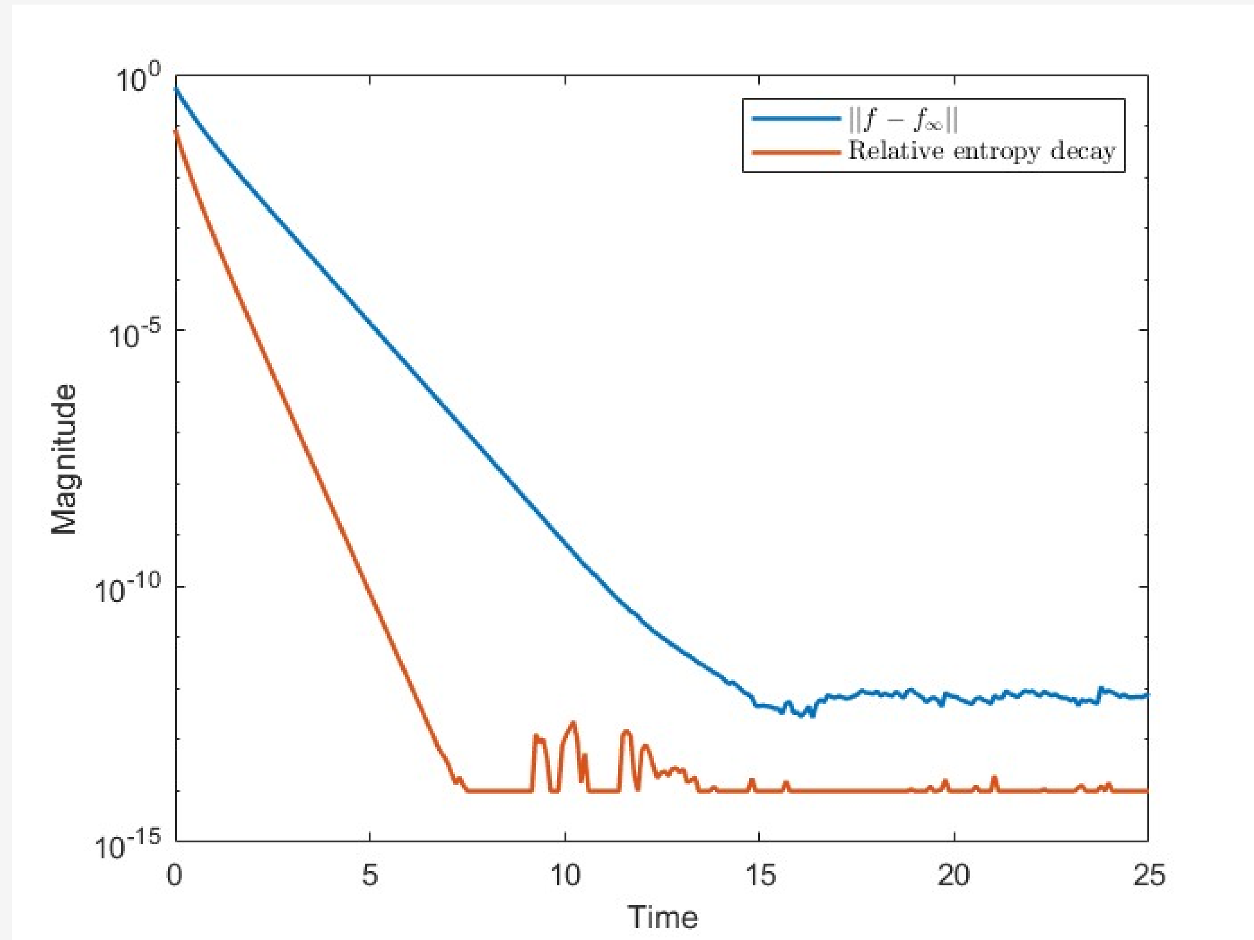
  - Solve for $\mathbf{K}^{k+1}, \mathbf{L}^{k+1}, \mathbf{S}^{k+1} \mapsto \mathbf{V}_\perp^{k+1}, \mathbf{V}_\parallel^{k+1}, \mathbf{S}^{k+1}$
  - Apply conservative truncation procedure [2].

Dougherty-Fokker-Planck Equation 0D2V

$$f_t = C_{\alpha\alpha}(f) = \frac{n_\alpha}{T_\alpha^{3/2}} \nabla_{\mathbf{v}} \cdot \left((\mathbf{v} - \mathbf{u}_\alpha) f + \frac{T_\alpha}{m_\alpha} \nabla_{\mathbf{v}} f\right)$$

$$f_{M1} + f_{M2} \to f_\infty$$

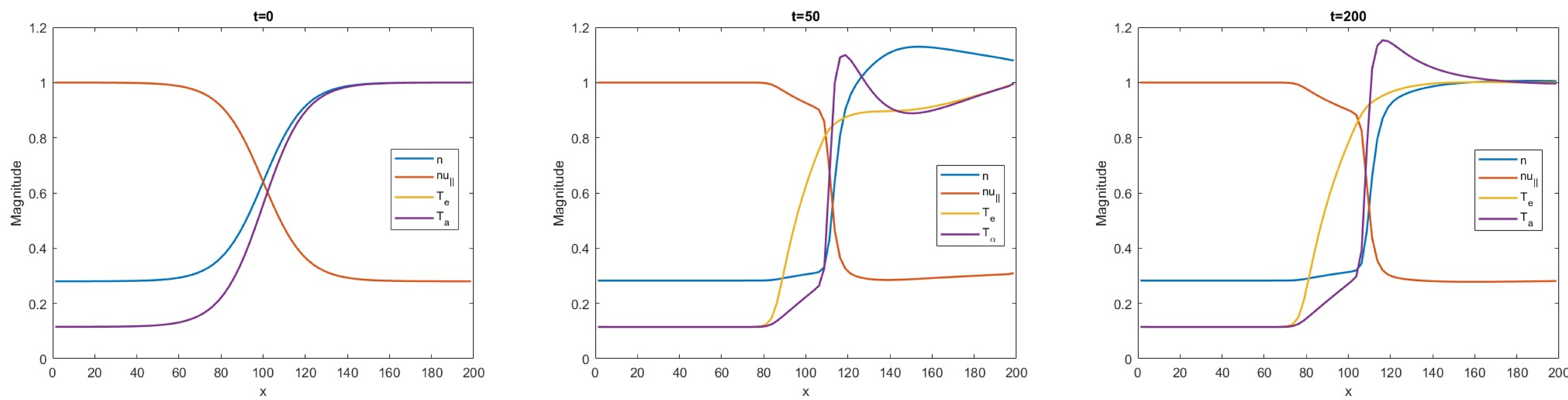
$$f_M(n, u_\perp, u_\parallel, T) = \frac{n}{(2\pi RT)^{3/2}} \exp\left(-\frac{(v_\perp - u_\perp)^2 + (v_\parallel - u_\parallel)^2}{2RT}\right)$$



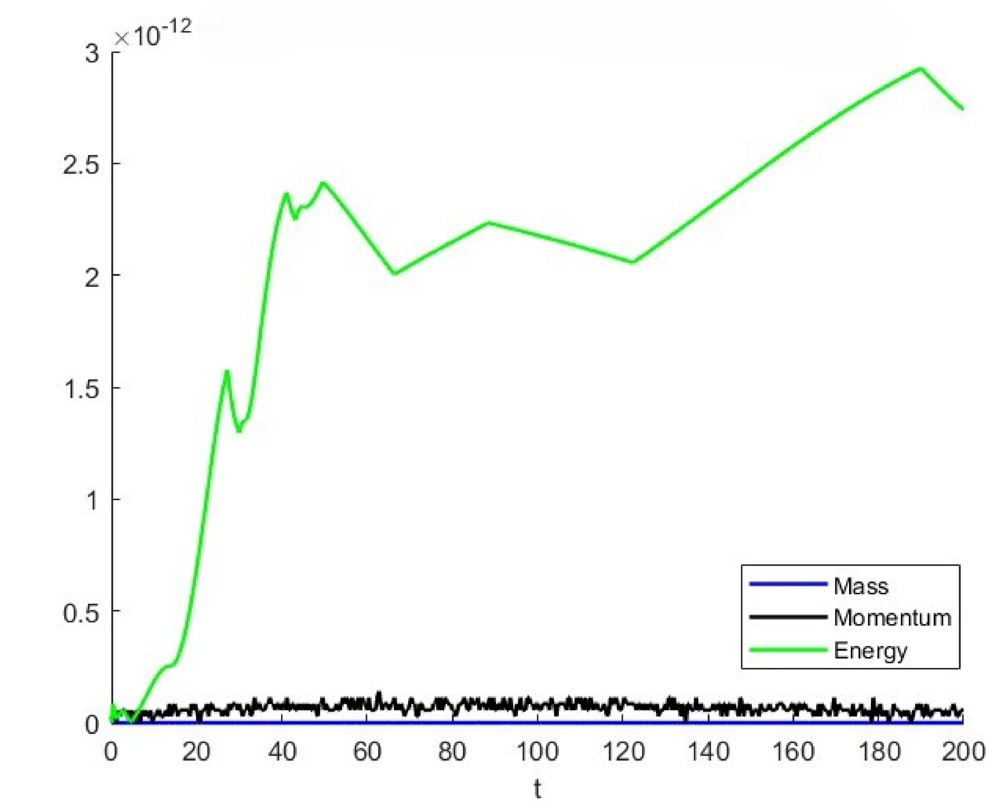
L_1 drive to equilibrium solution and relative entropy decay up to machine precision

References

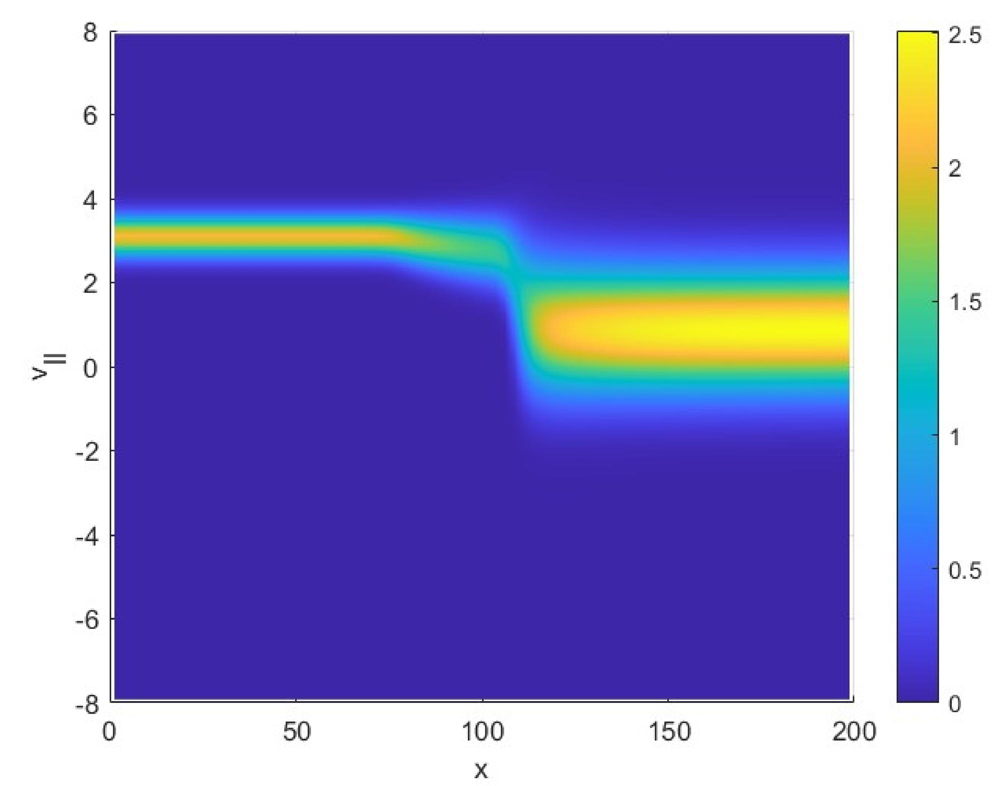
[1] Nakao, J., Qiu, J.M. and Einkemmer, L., 2025. Reduced Augmentation Implicit Low-rank (RAIL) integrators for advection-diffusion and Fokker–Planck models. SIAM Journal on Scientific Computing, 47(2), pp.A1145-A1169.
[2] Guo, W. and Qiu, J.M., 2024. A Local Macroscopic Conservative (LoMaC) low rank tensor method for the Vlasov dynamics. Journal of Scientific Computing, 101(3), p.61.



Evolution of moments n, $(nu_\parallel)$, T_α, and T_e at each spatial node x_i at times $t = 0, t = 50, t = 200$.



Mass, momentum, and energy conservation.



Numerical solution at $t = 200$ integrated along $v_\perp$.